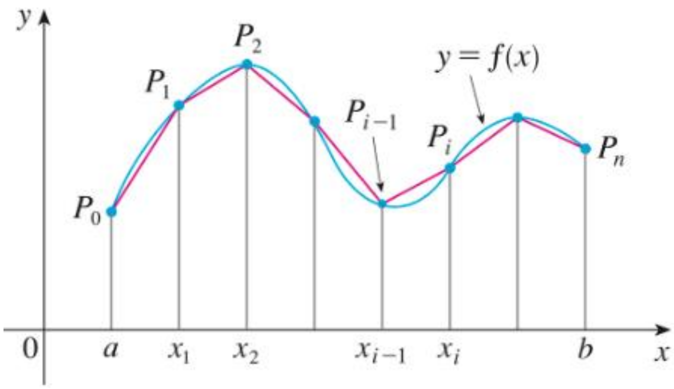
8.1 Arc Length

If given a simple shape and we wanted to find the distance around (perimeter) the shape, we would add all of the lengths of the sides of the shape to find the total length around the shape. For example, if the shape were a square, we would have 4 sides of equal length. The Perimeter of the square would be 4 times the length of one side, **n**, which would be **4n.**

Consider the following function, **y = f(x)**, with the following behavior.



If ***f*** is continuous on [a, b], we can obtain a polygonal approximation to the curve of **y = f(x)**, call it **C**, by dividing the interval [a, b] into **n** subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on **C** and the polygon with vertices $P_0, P_1, P_2, \ldots P_n$ is an approximation to **C**.

The length **L** of **C** is approximately the length of this polygon and the approximation gets better as **n** increases. We get the following:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}, P_i|$$

Where $|P_{i-1}, P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ (Note: Δx does not change whereas Δy does.)

Then apply the Mean Value Theorem to **f** on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\Delta x$$

Substitute this back into the equation for L and we get the formula for Arc Length:

The Arc Length Formula: If f' is continuous on [a, b], then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In Leibniz notation

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Find the length of the curve $y = f(x) = x^{\frac{3}{2}}$ between x = 0 and x = 4.

First find $\boldsymbol{f'(x)}$, $\quad \boldsymbol{f'(x) = \frac{3}{2}x^{\frac{1}{2}}}$

(Note that $\boldsymbol{f'(x)}$ is continuous on the interval [0, 4]. Using the arc length formula gives:

$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x\, dx}$$

Integrate using u-substitution: let $\mathbf{u} = 1 + \frac{9}{4}x$ thus $\mathbf{du} = \frac{9}{4}$dx when x = 0 → u = 1, when x = 4 → u = 10

$$(4/9\boldsymbol{du} = dx)$$

$$= \frac{4}{9}\int_1^{10} \sqrt{u}\, du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{10} = \frac{4}{9}\left[\frac{2}{3}\left(10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\right] = \frac{8}{27}\left(10^{\frac{3}{2}} - 1\right) \approx \boldsymbol{9.1\ units\ long}$$

Example: Find the length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval [0, ln 2]

$$f'(x) = 2e^x - \frac{1}{8}e^{-x} \quad \left(f(x)\right)^2 = \left(2e^x - \frac{1}{8}e^{-x}\right)^2 = \left(2e^x - \frac{1}{8}e^{-x}\right)\left(2e^x - \frac{1}{8}e^{-x}\right)$$

$$= 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$$

Substitute into the arc length formula:

$$L = \int_0^{\ln 2} \sqrt{1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx = \int_0^{\ln 2} \sqrt{4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}}\ \ dx = \int_0^{\ln 2} \sqrt{\left(2e^x + \frac{1}{8}e^{-x}\right)^2}\, dx$$

$$= \int_0^{\ln 2} \left(2e^x + \frac{1}{8}e^{-x}\right) dx = \left[2e^x - \frac{1}{8}e^{-x}\right]_0^{\ln 2}$$

$$= \left(2e^{\ln 2} - \frac{1}{8}e^{-\ln 2}\right) - \left(2e^0 - \frac{1}{8}e^0\right) = \boldsymbol{\frac{33}{16}}$$

Example: Find the length of the curve $y = f(x) = x^{\frac{2}{3}}$ between x = 0 and x = 8.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

Note: This function is undefined for **x = 0** therefore the arc length formula with respect to **x** cannot be used. BUT let's try describing the curve with respect to **y**. Solve $y = x^{\frac{2}{3}}$ for **x** and you get $x = \pm y^{\frac{3}{2}}$ When x = 0, y = 0 and when x = 8, y = 4. Since both y-values are positive, we can use just the positive portion of the function. $x = y^{\frac{3}{2}}$ and $x' = \frac{3}{2}y^{\frac{1}{2}}$.

$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2}y^{\frac{1}{2}}\right)^2}\, dy = \int_0^4 \sqrt{1 + \frac{9}{4}y}\ dy \approx \boldsymbol{9.1\ units\ of\ length}$$